MATH 3060 : HW 7 Solution

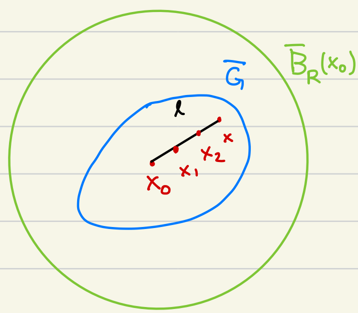
Leon Li

ylli@math.cuhk.edu.hk

1. Let G be a bounded convex open subset of $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in $C(\overline{G})$ if there exists a point $x_0 \in \overline{G}$ and a constant $M > 0$ such that $|f(x_0)| \le M$ for all f in the family.

Sol) Let $\mathcal{G} \subseteq C(\overline{G})$ be a family of equicontinuous functions satisfying the assumption. Since G is bounded, there exists $R > 0$ such that $\overline{G} \subseteq \overline{B}_R(x_0)$

Since $\mathcal{G}$ is equicontinuous, choose $\varepsilon = 1$, there exists $0 < \delta < R$ such that for any $x, y \in \overline{G}$ with $|x-y| < \delta$, for any $f \in \mathcal{G}$, $|f(x) - f(y)| < 1$.



Showing $\mathcal{G}$ is bounded: given any $f \in \mathcal{G}$ and $x \in \overline{G}$,

since $\overline{G}$ is convex, $l := \{(1-t)x_0 + tx \mid t \in [0,1]\} \subseteq \overline{G}$.

Dividing l into line segments with lengths lying over suitable range:

There exists $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that $x_i := (1-t_i)x_0 + t_i x \in l$ satisfies

$\frac{\delta}{2} < |x_i - x_{i-1}| < \delta$, for any $1 \le i \le n$.

Since $\overline{G} \subseteq \overline{B}_R(x_0)$, $R > |x_n - x_0| = \sum_{i=1}^{n} |x_i - x_{i-1}| > n \cdot \frac{\delta}{2}$, i.e. $n < \frac{2R}{\delta}$.

$\therefore$ $|f(x) - f(x_0)| \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < n \cdot 1 = n$, since $|x_i - x_{i-1}| < \delta$ for any $1 \leq i \leq n$.

$\therefore$ $|f(x)| \leq |f(x_0)| + |f(x) - f(x_0)| \leq M + n < M + \frac{2R}{\delta}$ is independent of f and x.

$\therefore$ $\mathcal{G}$ is bounded.

2. Show that the sequence

$$f_n(x) = \int_0^x \frac{\sqrt{n}}{\sqrt{n+t^3}} e^{-nt^2} dt, \quad x \in [0,1]$$

has a convergent subsequence in $(C[0,1], d_\infty)$.

Sol) Let $g_n(t) = \frac{\sqrt{n}}{\sqrt{n+t^3}} e^{-nt^2}$ for $t \in [0,1]$, $n \in \mathbb{N}$.

Showing $\{g_n\}_{n=1}^\infty$ is uniformly bounded:

For any $t \in [0,1]$, $n \in \mathbb{N}$, $|g_n(t)| = \frac{\sqrt{n}}{\sqrt{n+t^3}} e^{-nt^2} \leq 1 \cdot 1 = 1$.

$\therefore$ $\|g_n\|_\infty \leq 1$, for any $n \in \mathbb{N}$. Therefore, $\{g_n\}_{n=1}^\infty$ is uniformly bounded.

To show $\mathcal{G} := \{f_n \mid n \in \mathbb{N}\}$ has a convergent subsequence in $(C[0,1], d_\infty)$,

by Ascoli's Theorem, it suffices to show that $\mathcal{G}$ is equicontinuous and bounded:

① $\mathcal{G}$ is equicontinuous: for any $n \in \mathbb{N}$, $x, y \in [0,1]$,

$$|f_n(y) - f_n(x)| = \left|\int_0^y g_n(t)dt - \int_0^x g_n(t)dt\right| = \left|\int_x^y g_n(t)dt\right| \leq 1 \cdot |y-x| = |y-x|$$

$\therefore$ $\mathcal{G}$ is "uniformly Lipschitz continuous", hence is equicts by last example in Lecture 16.

② $\mathcal{G}$ is bounded: for any $n \in \mathbb{N}$, $x \in [0,1]$, $|f_n(x)| = \left|\int_0^x g_n(t)dt\right| \leq 1 \cdot |x| \leq 1$.

$\therefore$ By Ascoli's Theorem, $\mathcal{G}$ has a convergent subsequence in $(C[0,1], d_\infty)$.

3. Show that for any fixed $M > 0$, every sequence in

$$\mathscr{C}_M = \left\{ f \in C^1[0,1] : f(0) = 0 \text{ and } \int_0^1 |f'(t)|^2 dt \leq M \right\}$$

contains a convergent subsequence in $(C[0,1], d_\infty)$.

Sol) By Ascoli's Theorem, it suffices to show that $\mathscr{C}_M$ is equicts and bounded:

① $\mathscr{C}_M$ is equicontinuous: Similar idea as the proof of Prop. 4.1.

Given $f \in \mathscr{C}_M$, $x, y \in [0,1]$, without loss of generality assume $x < y$.

$$f(y) - f(x) = \int_0^1 \frac{d}{dt}\left(f((1-t)x + ty)\right) dt \quad \text{(by Fundamental Theorem of Calculus)}$$

$$= \left(\int_0^1 f'((1-t)x + ty)\, dt\right) \cdot (y - x)$$

$$\therefore |f(y) - f(x)| \leq \left(\int_0^1 |f'((1-t)x + ty)|\, dt\right) \cdot |y - x|$$

$$\leq \sqrt{\left(\int_0^1 |f'((1-t)x + ty)|^2 dt\right)} \cdot \sqrt{\int_0^1 dt} \cdot |y - x| \quad \text{(By Cauchy-Schwarz Inequality)}$$

$$= \sqrt{\left(\int_x^y |f'(s)|^2 \left(\frac{ds}{y-x}\right)\right)} \cdot |y - x| \quad (\text{Let } s = (1-t)x + ty,\ t \in [0,1])$$

$$\leq \sqrt{\left(\int_0^1 |f'(s)|^2 ds\right)} \cdot |y - x|^{\frac{1}{2}} \quad (\because [x,y] \subseteq [0,1])$$

$$\leq \sqrt{M} \cdot |y - x|^{\frac{1}{2}}$$

$\therefore$ $\mathscr{C}_M$ is "uniformly Hölder continuous", hence is equicts by last example in Lecture 16.

② $\mathcal{C}_M$ is bounded : By assumption, for any $f \in \mathcal{C}_M$, $f(0) = 0$.

Applying Q1 to $\mathcal{E} = \mathcal{C}_M$, which is equicontinuous by ①, with $G = (0,1)$, $x_0 = 0$, $M = 1$, it follows that $\mathcal{C}_M$ is bounded.

By Ascoli's Theorem, $\mathcal{C}_M$ is precompact in $(C[0,1], d_\infty)$,

i.e. every sequence in $\mathcal{C}_M$ contains a convergent subsequence in $(C[0,1], d_\infty)$

4. Suppose that $\sigma : [0, +\infty) \to \mathbb{R}$ is a continuous, nondecreasing function with $\sigma(0) = 0$. Show that

$$\mathcal{E}_\sigma = \{ f \in C[a,b] : |f(x) - f(y)| \leq \sigma(|x-y|), \forall x, y \in [a,b] \}$$

is an equicontinuous family.

Sol) Checking $\mathcal{C}_\sigma$ is equicontinuous by definition:

Given $\varepsilon > 0$, by continuity of σ at 0, there exists $\delta > 0$ such that

for any $0 \leq r < \delta$, $\varepsilon > |\sigma(r) - \sigma(0)| = |\sigma(r)|$ $(\because \sigma(0) = 0)$

$= \sigma(r)$ $(\because r \geq 0 \Rightarrow \sigma(r) \geq \sigma(0) = 0)$

$\therefore$ For any $f \in \mathcal{C}_\sigma$, for any $x, y \in [a,b]$ with $|x-y| < \delta$,

$|f(x) - f(y)| \leq \sigma(|x-y|) < \varepsilon$

$\therefore$ $\mathcal{C}_\sigma$ is equicontinuous.

Rmk Choosing $\sigma(r) := L r^\alpha$ for $L > 0$, $0 < \alpha \leq 1$ recovers the examples of family of "uniformly Hölder/Lipschitz-continuous" functions as in Lecture 16.